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DISCUSSION

DYNAMIC RESPONSE OF A MULTI-LAYERED POROELASTIC MEDIUM[1]

DISCUSSION BY EDUARDO KAUSEL

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The authors have presented a most authoritative and interesting study on the propagation of waves in poroelastic media, in which they included also a thorough review of past work on the subject.[1] Thus, the purpose of this discussion is only to rectify some minor misconceptions, and to add a few relevant comments.

In the introduction, the authors classify the solution tools for wave motion in layered media into *rigorous* analytical formulations (Thomson-Haskell, etc.), and *approximate* methods rooted in finite element concepts (Lysmer-Waas, etc.), References 14–23 in the authors' paper.[1] As briefly mentioned in the paper, the latter methods rely on a discretization of the medium in the direction of layering (i.e. the vertical), so it seems natural to refer to them generically as the *Thin Layer Method*,[2] a term which we shall use and abbreviate in the following as TLM. Among previous works based on the TLM, the authors also included one of our papers,[3] which we shall abbreviate as K&R. However, this paper included both the *exact* layer stiffness matrices for a viscoelastic—valid for arbitrarily thick layers, including a half-space—and the *discrete* ones based on thin layers. Hence, the authors' rigidity matrices are not only conceptually similar to those in our paper, but should in fact reduce to those in our Tables I–V when the porosity of the medium tends to zero (i.e. for a purely viscoelastic medium).

While referring to the TLM, the writers state (italics mine) "Numerical solutions of the global stiffness equations ... and subsequent application of *numerical quadrature* to evaluate integrals over the wave number domain yields the response". They then add: "In addition to the approximate nature of the solution, this method has the disadvantage that the presence of an *underlying half-space* cannot be taken into consideration in a consistent manner". We beg to disagree. Firstly, the most significant—indeed extraordinary—advantage of the TLM over the exact stiffness matrix method is that one can dispose of either the integration over wave numbers,[4] or that over frequencies.[2,5] Thus, those integral transforms can be evaluated *exactly*, do not require numerical quadrature, and are *not* subjected to spatial/temporal aliasing or the numerical errors associated with either truncation or the large and rapid oscillations of the kernels (particularly when the source and receiver are at the same elevation). Secondly, in the TLM a half-space can indeed be modelled consistently and with excellent results by means of well-known paraxial approximations;[6,7] this fact is demonstrated in Figure 1 below, which shows a comparison between the exact solution to Stokes' problem (a harmonic point load within a homogeneous full space) and that obtained with the TLM with paraxial boundaries. In the latter, the full space is modelled with a region ('sandwich') of homogeneous layers to which paraxial boundaries are added at the upper and lower boundaries. The agreement is nearly perfect.

On a related matter, when carrying out the improper integrals over wave number and frequency with the exact stiffness matrices, it is advisable to use complex frequencies to avoid the large oscillations in the kernels, particularly for lightly damped (or even undamped) media. This method is often referred to in the signal processing literature as the Exponential Window Method (EWM). The imaginary component of frequency is usually chosen to be either slightly smaller or equal to the frequency step, the effect of which is roughly

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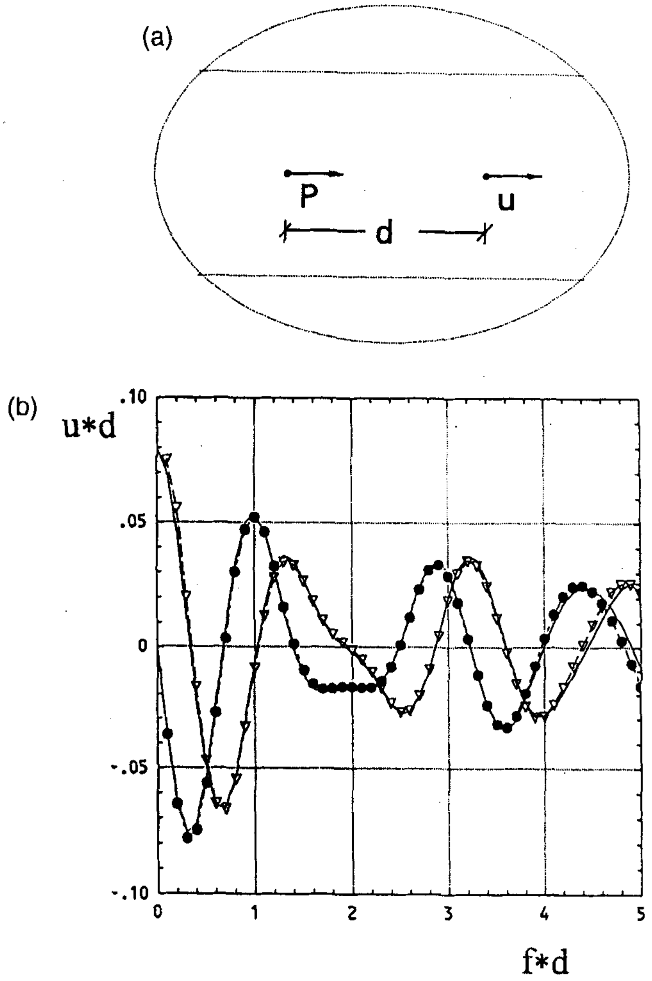

Figure 1. Stokes' problem: (a) dynamically loaded full space; (b) horizontal displacements, Poisson's ratio = 0·20

equivalent to mass-proportional damping. However, this fictitious damping is later removed completely from the solution by means of a rising exponential window.[8,9] This strategy practically eliminates the spatial/temporal aliasing in the computed dynamic signatures.

We observe that the authors multiply one of the displacement and traction (stress) components by $i = \sqrt{-1}$ so as to achieve *symmetric* stiffness matrices, a strategy that we first proposed in K&R. While the authors chose to apply this factor to the horizontal components $\bar{u}_x, \bar{\sigma}_{zx}$ in K&R, we instead modified the vertical components $\bar{u}_z, \bar{\sigma}_{zz}$. In principle, this difference should lead to only a trivial checkerboard style change in the signs of the (plane-strain) stiffness elements, but the choice in K&R seems to have one additional advantage: when the stiffness matrices for the layers are reformulated in cylindrical co-ordinates, it is found that they are identical to a combination of the matrices for the two plane-strain cases (SV-P and SH, respectively). Presumably, to achieve this same advantage when extending the authors' matrices to cylindrical co-ordinates would require multiplying *both* horizontal components u_x, u_y by i; conceivably, this could be complicated by the presence of the additional components in fluid pressure and relative fluid motion.

We agree with the authors that the stiffness matrix method offers significant advantages over the classical propagator/transfer matrix method, since in comparison to the latter, it involves (a) half the number of degrees of freedom; (b) half the bandwidth; (c) the matrices are symmetric; (d) no numerical instability

problems arise as a result of thick layers or large frequencies; and (e) it avoids terms of exponential growth. Indeed, the stiffness matrices are well conditioned when using complex frequencies or the material has hysteretic damping. However, some caution is necessary when considering nearly undamped systems *and* the EWM is *not* used, because the matrices may then become ill-conditioned at the (physical or purely mathematical) cut-off frequencies of the individual layers. To illustrate this point, consider the simplest possible stiffness matrix, namely that for SH waves propagating *vertically* in a *single* homogeneous horizontal layer. As given in K&R Table V, this matrix is

$$\mathbf{K} = \frac{\rho C_s \omega}{\sin \eta}\begin{Bmatrix} \cos \eta & -1 \\ -1 & \cos \eta \end{Bmatrix}$$

where ρ is the mass density, C_s the shear wave velocity; ω the frequency and $\eta = \omega h / C_s$, with h being the thickness of the layer. The determinant of this matrix is

$$\det \mathbf{K} = \frac{(\rho C_s \omega)^2}{\sin^2 \eta}(\cos^2 \eta - 1) = -(\rho C_s \omega)^2$$

which changes quadratically with frequency and is always non-zero for non-zero frequencies. However, the determinant of the terms enclosed by the braces becomes zero when $\cos^2 \eta = 1$, that is, when $\eta = k\pi$, or $\omega = k\pi C_s/h$, which are the resonant frequencies for a single free layer. Hence, we obtain the most remarkable result that at the resonant frequencies of the layer, the stiffness matrix is singular, yet its determinant is not zero! Of course, the reason for this anomaly is that at these frequencies, the stiffness elements themselves become infinitely large. Thus, a determinant search procedure for finding the resonant frequencies of such a system will surely fail, with little indication of ill-conditioning. This result remains valid when considering non-zero horizontal wave numbers, in which case the stiffness matrices become singular at the layer's so-called cutoff frequencies.

More generally, consider what happens when two (or more) layers with different (or even the *same*) material properties or thicknesses are joined and their dynamic stiffness matrices are overlapped so as to form the rigidity matrix of the system of layers: the elements of each matrix become singular at the layers' respective *mathematical* cut-off frequencies, even though these frequencies are not *system* frequencies anymore. Thus, numerical difficulties can be expected to occur in the neighbourhood of these mathematical cutoff frequencies. These problem are largely (or completely) avoided by using the EWM. On the other hand, this problem points to potential difficulties in using the exact stiffness matrix method to find the cutoff frequencies of layered systems by means of search techniques. This problem is also avoided in the TLM, since the elements are then never singular. Thus, we agree that the exact stiffness matrix method offers many advantages over the classical methods, but care is still required in their use.

REFERENCES

1. R. K. N. D. Rajapakse and T. Senjuntichai, 'Dynamic response of a multi-layered poroelastic medium', *Earthquake eng. struct. dyn.* **24**, 703–722 (1995).
2. E. Kausel, "Discrete Green's functions for layered media: time domain solution', *Proc. of the NSF workshop geophysical techniques for site and material characterization*, June 11–12, 1993, Atlanta, Georgia, 1993.
3. E. Kausel and J. M. Roesset, 'Stiffness matrices for layered soils', *Bull. seism. soc. Am.* **71**, 1743–1761 (1981).
4. E. Kausel and R. Peek, 'Dynamic loads in the interior of a layered stratum: an explicit solution', *Bull. seism. soc. Am.* **72**, 1459–1481 (1982).
5. E. Kausel, 'Thin-layer method: formulation in the time domain, *Int. j. numer. methods eng.* **37**, 927–941 (1994).
6. S. H. Seale and E. Kausel, 'Point loads in cross-anisotropic, layered half-spaces', *J. eng. mech. ASCE* **115**, 509–524 (1989).
7. Hull (Seale), S. and E. Kausel, 'Dynamic loads in layered halfspaces', *Proc 5th engineering mechanics division specialty conf. ASCE*, Vol. I, Laramie, Wyoming, August 1984, pp. 201–204.
8. R. A. Phinney, 'Theoretical calculation of the spectrum of first arrivals in layered elastic mediums', *J. geophys. res.* **70**, 5107–5123 (1965).
9. E. Kausel and J. M. Roësset, 'Frequency domain analysis of undamped systems, *J. eng. mech. ASCE* **118**, 721–734 (1992).
10. E. Kausel, J. M. Roësset and G. Bouckovalas, 'A stiffness matrix approach for layered soils', *Proc. 7th world conf. on earthquake engineering*, Istanbul, Turkey, Vol. 3, September 1980.